

MIXED PENTAGON, OCTAGON AND BROADHURST DUALITY EQUATION

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ABSTRACT. This paper is on elimination of defining equations of the cyclotomic analogues, introduced by the first author, of Drinfeld's scheme of associators. We show that the mixed pentagon equation implies the octagon equation for $N = 2$ and the particular distribution relation. We also explain that Broadhurst duality is compatible with the torsor structure. We develop a formalism of infinitesimal module categories and use it for deriving a proof left implicit in the first named author's earlier work.

CONTENTS

0. Introduction	1
1. The Grothendieck-Teichmüller group	3
2. The cyclotomic Grothendieck-Teichmüller group	5
3. Mixed pentagon and octagon equations	10
4. Broadhurst duality	14
Appendix A. Infinitesimal module categories	15
A.1. Infinitesimal braided monoidal categories	15
A.2. Infinitesimal module categories over C_N -braided monoidal categories	17
A.3. Automorphisms	18
Appendix B. Erratum of [E]	19
References	19

0. INTRODUCTION

The “Grothendieck-Teichmüller theory” was developed by Drinfeld [Dr] with the motivation of quantization of certain Hopf algebras related with the monodromy of the KZ differential system (Kohno-Drinfeld theorem), and in close relation with Grothendieck's approach to the description of the action of the absolute Galois group of the rational number field \mathbf{Q} on the “Teichmüller tower” ([G]). One of the main results of this theory is a collection of relations between periods of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ called the MZV's (multiple zeta values) (see (1.10)). These relations are derived from the study of the monodromy of the KZ system, and fall in three classes : two classes of hexagon (1.9) and one class of pentagon relations (1.7). The elimination of the hexagon relations (i.e., the statement that they are consequences of the pentagon relation) was established by the second-named author in [F10a] (see theorem 1.6 below). The proof uses combinatorial arguments based on the cell decomposition of the compactification of the moduli space $\mathfrak{M}_{0,5}$.

The Grothendieck-Teichmüller theory was extended in the cyclotomic context by the first-named author [E]. In this theory, an integer $N \geq 1$ is fixed and the analogues of the MZV's are periods of the motivic fundamental group the algebraic curve $\mathbf{P}^1 \setminus \{0, \mu_N, \infty\}$ (μ_N : the group of N -th roots of unity) called multiple L -values (see (2.12)). The study of the monodromy of the cyclotomic KZ system yields a collection of relations between these numbers. It is shown that the pro-algebraic variety $\text{Psdist}(N, \mathbf{k})$ over \mathbf{k} (\mathbf{k} : a field of characteristic 0) arising from the ‘cyclotomic KZ’ relations is equipped with a torsor structure over pro- \mathbf{k} -algebraic group $\text{GRTMD}(N, \mathbf{k})$, which is the extension of $(\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{G}_m$ by a pronipotent \mathbf{k} -algebraic group, and whose Lie \mathbf{k} -algebra $\text{grtm}\mathfrak{d}(N, \mathbf{k})$ is non-negatively graded. Another family of relations between MZV's, the ‘double shuffle and regularization relations’ were studied in [IKZ, R]; cyclotomic analogues of these relations were discussed in [R]. In [F11, F10b], it was proved that these relations are consequences of the ‘KZ’ and ‘cyclotomic KZ’ relations.

For particular values of N , exceptional symmetries of $\mathbf{P}^1 \setminus \{0, \infty, \mu_N\}$ give rise to additional families of relations. When $N = 2$, these relations were made explicit by Broadhurst ([Bh]), and for $N=4$, by Okuda ([O]); Okuda's relations allow one to recover Broadhurst's (see [O] §4). The results of this paper are of four types :

- (A) elimination results for ‘cyclotomic KZ’ relations between multiple L -values
- (B) elimination of defining conditions for pro-algebraic groups and Lie algebras
- (C) insertion of Broadhurst's result in the framework of torsors
- (D) a theory of infinitesimal module categories, leading to a proof of a result left implicit in [E], and which is also used in the proof of (B).

We now explain these results in more detail.

(A). The ‘cyclotomic KZ’ relations fall in the following classes : the mixed pentagon (2.5), octagon (2.9) and distribution (2.10) classes ; there is one class of distribution relations for each N' dividing N , $N' \neq N$. Our first result is the implication of the first distribution relation from the mixed pentagon equation:

Theorem 0.1 (Proposition 2.9). *If a pair of two group-like elements $(g, h) \in \exp \mathfrak{t}_3^0 \times \exp \mathfrak{t}_{3,N}^0$ (for the notations see below) satisfies the mixed pentagon equation (2.5), then it also satisfies the distribution relation (2.10) for $N' = 1$.*

As a consequence, we obtain the equality of two groups

$$\text{GRTMD}_{(\bar{1},1)}(N, \mathbf{k}) = \text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})$$

and of two torsors

$$\text{Psdist}_{(\bar{1},1)}(N, \mathbf{k}) = \text{Pseudo}_{(\bar{1},1)}(N, \mathbf{k})$$

for a prime N (see Corollary 2.11).

(B). The Lie algebras $\text{grtm}\mathfrak{d}(N, \mathbf{k})$ are defined by mixed pentagon (2.1), octagon (2.2), speciality (2.3) and distribution (2.10) equations. It was proved in [E] that the speciality condition implies the octagon one. We prove that for $N = 2$ the mixed pentagon equation implies the octagon equation:

Theorem 0.2 (Theorem 3.1). *For $N = 2$ if a pair of two Lie elements $(\varphi, \psi) \in \mathfrak{t}_3^0 \times \mathfrak{t}_{3,2}^0$ with $c_{B(0)}(\psi) = c_{AB(0)}(\psi) = 0$ (for the notations see below) satisfies the mixed pentagon equation (2.1), then it also satisfies the octagon equation (2.2).*

While this result may be viewed as not particularly useful in view of the result of [E], its proof might be of interest as it extends the combinatorial arguments of [F10a] to a Kummer covering $\tilde{\mathfrak{M}}_{0,5}^2$ of the moduli space $\mathfrak{M}_{0,5}$.

The pro-algebraic groups $GRTMD(N, \mathbf{k})$ are similarly defined by the mixed pentagon (2.5), octagon (2.6) and speciality equations (2.7). We prove that for $N = 2$ the octagon equation is implied by the other two equations in the setting:

Theorem 0.3 (Theorem 3.4). *For $N = 2$ if a pair of two group-like element $(g, h) \in \exp \mathfrak{t}_3^0 \times \exp \mathfrak{t}_{3,2}^0$ with $c_{B(0)}(h) = c_{AB(0)}(h) = 0$ satisfies the mixed pentagon equation (2.5) and the special action condition (2.7), then it also satisfies the octagon equation (2.6).*

(C). In [Bh], Broadhurst introduced a family of ‘duality’ relations among multiple L -values for $N = 2$. These relations will be shown to be compatible with the torsor structure of $\text{Psdist}_{(\bar{1},1)}(2, \mathbf{k})$:

Theorem 0.4 (Theorem 4.2). *The subset $\text{PseudoB}_{(\bar{1},1)}(2, \mathbf{k})$ defined by the Broadhurst duality (4.2) forms a subtorsor of $\text{Psdist}_{(\bar{1},1)}(2, \mathbf{k})$.*

(D). The notion of infinitesimal module categories over braided monoidal categories is introduced in our appendix. It is defined by several axioms including the mixed pentagon axiom. In Proposition A.2 the notion is employed to prove that the set $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})$ forms a group by the multiplication (2.8).

The structure of the paper is the following. §1 and §2 are a review of the Grothendieck-Teichmüller theory in [Dr] and [E]. In §2, elimination result (A) is proved (Proposition 2.9). Results (B) on mixed pentagon and octagon relations are proved in §3 (Theorems 3.1 and 3.4). Result (C) on compatibility of the Broadhurst duality relations with a torsor structure is proved in §4 (Theorem 4.2). Appendix A contains result (D), i.e., the basics of infinitesimal module category and the proof of the fact implicitly used in [E] that $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})$ is a group. Some errors in [E] are corrected in Appendix B.

1. THE GROTHENDIECK-TEICHMÜLLER GROUP

This section is a short review on Drinfeld’s theory of associators in [Dr].

Let \mathbf{k} be a field of characteristic 0. For $n \geq 2$, the Lie algebra \mathfrak{t}_n of infinitesimal pure braids is the completed \mathbf{k} -Lie algebra with generators t^{ij} ($i \neq j$, $1 \leq i, j \leq n$) and relations

$$t^{ij} = t^{ji}, [t^{ij}, t^{ik} + t^{jk}] = 0 \text{ and } [t^{ij}, t^{kl}] = 0 \text{ for all distinct } i, j, k, l.$$

We note that \mathfrak{t}_2 is the 1-dimensional abelian Lie algebra generated by t^{12} . The element $z_n = \sum_{1 \leq i < j \leq n} t^{ij}$ is central in \mathfrak{t}_n . Put \mathfrak{t}_n^0 to be the Lie subalgebra of \mathfrak{t}_n with the same generators except t^{1n} and the same relations as \mathfrak{t}_n . Then we have

$$\mathfrak{t}_n = \mathfrak{t}_n^0 \oplus \mathbf{k} \cdot z_n.$$

When $n = 3$, \mathfrak{t}_3^0 is the free Lie algebra \mathfrak{F}_2 of rank 2 with generators $A := t^{12}$ and $B := t^{23}$.

If S and T are two sets, then a *partially defined map* $f : S \rightarrow T$ means the data of (a) a subset $D_f \subset S$, and (b) a map $f : D_f \rightarrow T$. For a partially defined map $f : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$, the Lie algebra morphism $\mathfrak{t}_n \rightarrow \mathfrak{t}_m$, $x \mapsto x^f = x^{f^{-1}(1), \dots, f^{-1}(n)}$ is uniquely defined by

$$(t^{ij})^f = \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} t^{i'j'}.$$

Definition 1.1 ([Dr]). The Grothendieck-Teichmüller Lie algebra $\mathbf{grt}_1(\mathbf{k})$ is defined to be the set of $\varphi = \varphi(A, B) \in \mathfrak{t}_3^0$ satisfying the *duality and hexagon equations* in \mathfrak{t}_3^0

$$(1.1) \quad \varphi(A, B) + \varphi(B, A) = 0, \quad \varphi(A, B) + \varphi(B, C) + \varphi(C, A) = 0$$

with $A + B + C = 0$, the *special derivation condition* in \mathfrak{t}_3^0

$$(1.2) \quad [B, \varphi(A, B)] + [C, \varphi(A, C)] = 0,$$

and the *pentagon equation* in \mathfrak{t}_4^0

$$(1.3) \quad \varphi^{1,2,34} + \varphi^{12,3,4} = \varphi^{2,3,4} + \varphi^{1,23,4} + \varphi^{1,2,3}.$$

It actually forms a Lie algebra with the Lie bracket given by

$$(1.4) \quad \langle \varphi_1, \varphi_2 \rangle = [\varphi_1, \varphi_2] + D_{\varphi_2}(\varphi_1) - D_{\varphi_1}(\varphi_2),$$

where D_φ is the derivation of \mathfrak{t}_3^0 defined by

$$D_\varphi(A) = [\varphi, A] \quad \text{and} \quad D_\varphi(B) = 0.$$

The Lie algebra structure is realised by the embedding

$$\mathbf{grt}_1(\mathbf{k}) \hookrightarrow \text{Der}(\mathfrak{t}_3^0)$$

sending $\varphi \mapsto D_\varphi$.

Definition 1.2 ([Dr]). The Grothendieck-Teichmüller group $\mathbf{GRT}_1(\mathbf{k})$ is defined to be the set of series $g \in \exp \mathfrak{t}_3^0$ satisfying the *duality and hexagon equations* in $\exp \mathfrak{t}_3^0$

$$(1.5) \quad g(A, B)g(B, A) = 1, \quad g(C, A)g(B, C)g(A, B) = 1$$

with $A + B + C = 0$, the *special action condition* in $\exp \mathfrak{t}_3^0$

$$(1.6) \quad A + g(A, B)^{-1}Bg(A, B) + g(A, C)^{-1}Cg(A, C) = 0,$$

and the *pentagon equation* in $\exp \mathfrak{t}_4^0$

$$(1.7) \quad g^{1,2,34}g^{12,3,4} = g^{2,3,4}g^{1,23,4}g^{1,2,3}.$$

It forms a group by the multiplication

$$(1.8) \quad g_1 \circ g_2 = g_2(A, B) \cdot g_1(A, g_2^{-1}Bg_2).$$

The group structure is realised by the embedding (but anti-homomorphism)

$$\mathbf{GRT}_1(\mathbf{k}) \hookrightarrow \text{Aut} \mathfrak{t}_3^0$$

sending g to the automorphism A_g defined by

$$A \mapsto A \quad \text{and} \quad B \mapsto g^{-1}Bg.$$

We note that its associated Lie algebra is $\mathbf{grt}_1(\mathbf{k})$.

Definition 1.3 ([Dr]). The associator set $\mathbf{M}_1(\mathbf{k})$ is defined to be the set of series $g \in \exp \mathfrak{t}_3^0$ satisfying the pentagon equation (1.7) and the following variant of hexagon equations

$$(1.9) \quad g(A, B)g(B, A) = 1, \quad \exp\left\{\frac{A}{2}\right\}g(C, A)\exp\left\{\frac{C}{2}\right\}g(B, C)\exp\left\{\frac{B}{2}\right\}g(A, B) = 1$$

with $A + B + C = 0$.

It forms a right $\mathbf{GRT}_1(\mathbf{k})$ -torsor by (1.8) with $g_1 \in \mathbf{M}_1(\mathbf{k})$ and $g_2 \in \mathbf{GRT}_1(\mathbf{k})$.

Remark 1.4. It is shown that the special derivation condition (1.2) for $\varphi \in \mathfrak{t}_3^0$ (resp.(1.6) for $g \in \exp \mathfrak{t}_3^0$) follows from duality and hexagon equations (1.1) and the pentagon equation (1.3) in [Dr] proposition 5.7. (resp.(1.5) and (1.7) in [Dr] proposition 5.9.)

Remark 1.5. A typical example of elements in $M_1(\mathbf{C})$ is

$$\varphi_{KZ}(A, B) = \Phi_{KZ}\left(\frac{A}{2\pi\sqrt{-1}}, \frac{B}{2\pi\sqrt{-1}}\right)$$

constructed in [Dr], where $\Phi_{KZ}(A, B)$ is the *Drinfeld associator*. This series has the following expression:

$$\Phi_{KZ}(A, B) = 1 + \sum (-1)^m \zeta(k_1, \dots, k_m) A^{k_m-1} B \dots A^{k_1-1} B + (\text{regularized terms})$$

where $\zeta(k_1, \dots, k_m)$ are *multiple zeta values* defined by the following series

$$(1.10) \quad \zeta(k_1, \dots, k_m) = \sum_{0 < n_1 < \dots < n_m} \frac{1}{n_1^{k_1} \dots n_m^{k_m}}$$

for $m, k_1, \dots, k_m \in \mathbf{N}(= \mathbf{Z}_{>0})$ with $k_m \neq 1$ and for the regularised terms see [F03b].

For a monic monomial W in $U\mathfrak{t}_3^0 = \mathbf{k}\langle\langle A, B \rangle\rangle$, $c_W(g)$ for $g \in U\mathfrak{t}_3^0$ means the coefficient of W in g . On pentagon and hexagon equations we have the following.

Theorem 1.6 ([F10a]). (1). Let φ be an element of \mathfrak{t}_3^0 with $c_B(\varphi) = c_{AB}(\varphi) = 0$. If φ satisfies the pentagon equation (1.3), then it also satisfies duality and hexagon equations (1.1).

(2). Let g be an element of $\exp \mathfrak{t}_3^0$ with $c_B(g) = c_{AB}(g) = 0$. If g satisfies the pentagon equation (1.7), then it also satisfies duality and hexagon equations (1.5).

(3). Let g be an element of $\exp \mathfrak{t}_3^0$ with $c_B(g) = 0$ and $c_{AB}(g) \in \mathbf{k}^\times$. If g satisfies the pentagon equation (1.7), then the duality and hexagon equations (1.9) hold for $g(\frac{A}{\mu}, \frac{B}{\mu})$ with $\mu = \pm \sqrt{24c_{AB}(g)} \in \bar{\mathbf{k}}$.

Remark 1.7. In [F11] it is shown that the pentagon equation (1.7) implies the double shuffle relation and the regularization relation, which are one of the fundamental relations among multiple zeta values.

2. THE CYCLOTOMIC GROTHENDIECK-TEICHMÜLLER GROUP

This section is a review of the first named author's theory on the cyclotomic analogues of associators in [E].

Here we recall the notations¹ in [E]: For $n \geq 2$ and $N \geq 1$, the Lie algebra $\mathfrak{t}_{n,N}$ is the completed \mathbf{k} -Lie algebra with generators

$$t^{1i} \ (2 \leq i \leq n), \text{ and } t(a)^{ij} \ (i \neq j, 2 \leq i, j \leq n, a \in \mathbf{Z}/N\mathbf{Z})$$

and relations

$$\begin{aligned} t(a)^{ij} &= t(-a)^{ji}, \\ [t(a)^{ij}, t(a+b)^{ik} + t(b)^{jk}] &= 0, \\ [t^{1i} + t^{1j} + \sum_{c \in \mathbf{Z}/N\mathbf{Z}} t(c)^{ij}, t(a)^{ij}] &= 0, \end{aligned}$$

¹ Several of them are changed for our convenience.

$$[t^{1i}, t^{1j} + \sum_{c \in \mathbf{Z}/N\mathbf{Z}} t(c)^{ij}] = 0,$$

$$[t^{1i}, t(a)^{jk}] = 0 \text{ and } [t(a)^{ij}, t(b)^{kl}] = 0$$

for all $a, b \in \mathbf{Z}/N\mathbf{Z}$ and all distinct i, j, k, l ($2 \leq i, j, k, l \leq n$).

We note that $\mathfrak{t}_{n,1}$ is equal to \mathfrak{t}_n for $n \geq 2$. We have a natural injection $\mathfrak{t}_{n-1,N} \hookrightarrow \mathfrak{t}_{n,N}$. The Lie subalgebra $\mathfrak{f}_{n,N}$ of $\mathfrak{t}_{n,N}$ generated by t^{1n} and $t(a)^{in}$ ($2 \leq i \leq n-1$, $a \in \mathbf{Z}/N\mathbf{Z}$) is free of rank $(n-2)N+1$ and forms an ideal of $\mathfrak{t}_{n,N}$. Actually it shows that $\mathfrak{t}_{n,N}$ is a semi-direct product of $\mathfrak{f}_{n,N}$ and $\mathfrak{t}_{n-1,N}$. The element $z_{n,N} = \sum_{1 \leq i < j \leq n} t^{ij}$ with $t^{ij} = \sum_{a \in \mathbf{Z}/N\mathbf{Z}} t(a)^{ij}$ ($2 \leq i < j \leq n$) is central in $\mathfrak{t}_{n,N}$. Put $\mathfrak{t}_{n,N}^0$ to be the Lie subalgebra of $\mathfrak{t}_{n,N}$ with the same generators except t^{1n} and the same relations as $\mathfrak{t}_{n,N}$. Then we have

$$\mathfrak{t}_{n,N} = \mathfrak{t}_{n,N}^0 \oplus \mathbf{k} \cdot z_{n,N}.$$

Especially when $n = 3$, $\mathfrak{t}_{3,N}^0$ is free Lie algebra \mathfrak{F}_{N+1} of rank $N+1$ with generators $A := t^{12}$ and $B(a) = t(a)^{23}$ ($a \in \mathbf{Z}/N\mathbf{Z}$).

For a partially defined map $f : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ such that $f(1) = 1$, the Lie algebra morphism $\mathfrak{t}_{n,N} \rightarrow \mathfrak{t}_{m,N}$: $x \mapsto x^f = x^{f^{-1}(1), \dots, f^{-1}(n)}$ is uniquely defined by

$$(t(a)^{ij})^f = \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} t(a)^{i'j'} \quad (i \neq j, 2 \leq i, j \leq n)$$

and

$$(t^{1j})^f = \sum_{j' \in f^{-1}(j)} t^{1j'} + \frac{1}{2} \sum_{j', j'' \in f^{-1}(j)} \sum_{c \in \mathbf{Z}/N\mathbf{Z}} t(c)^{j'j''} + \sum_{i' \neq 1 \in f^{-1}(1), j' \in f^{-1}(j)} \sum_{c \in \mathbf{Z}/N\mathbf{Z}} t(c)^{i'j'}$$

($2 \leq j \leq n$). Again for a partially defined map $g : \{2, \dots, m\} \rightarrow \{1, \dots, n\}$, the Lie algebra morphism $\mathfrak{t}_n \rightarrow \mathfrak{t}_{m,N}$: $x \mapsto x^g = x^{g^{-1}(1), \dots, g^{-1}(n)}$ is uniquely defined by

$$(t^{ij})^g = \sum_{i' \in g^{-1}(i), j' \in g^{-1}(j)} t(0)^{i'j'} \quad (i \neq j, 1 \leq i, j \leq n).$$

Definition 2.1 ([E]). For $N \geq 1$, the Lie algebra $\mathbf{grtm}_{(\bar{1},1)}(N, \mathbf{k})$ is defined to be the set of pairs $(\varphi, \psi) \in \mathfrak{t}_3^0 \times \mathfrak{t}_{3,N}^0$ satisfying $\varphi \in \mathbf{grt}_1(\mathbf{k})$, the *mixed pentagon equation* in $\mathfrak{t}_{4,N}^0$

$$(2.1) \quad \psi^{1,2,34} + \psi^{12,3,4} = \varphi^{2,3,4} + \psi^{1,23,4} + \psi^{1,2,3},$$

the *octagon equation* in $\mathfrak{t}_{3,N}^0$

$$(2.2) \quad \begin{aligned} & \psi(A, B(0), B(1), \dots, B(i), \dots, B(N-1)) \\ & - \psi(A, B(1), B(2), \dots, B(i+1), \dots, B(0)) \\ & + \psi(C, B(1), B(0), \dots, B(N+1-i), \dots, B(2)) \\ & - \psi(C, B(0), B(N-1), \dots, B(N-i), \dots, B(1)) = 0 \end{aligned}$$

with $A + \sum_{a \in \mathbf{Z}/N\mathbf{Z}} B(a) + C = 0$,
the *special derivation condition* in $\mathfrak{t}_{3,N}^0$

$$(2.3) \quad \sum_{a \in \mathbf{Z}/N\mathbf{Z}} \left[\psi(A, B(a), B(a+1), \dots, B(a+i), \dots, B(a-1)), B(a) \right] \\ + \left[\psi(A, B(0), B(1), \dots, B(i), \dots, B(N-1)) \right. \\ \left. - \psi(C, B(0), B(N-1), \dots, B(N-i), \dots, B(1)), C \right] = 0$$

and $c_{B(0)}(\psi) = 0$.²

Here for any \mathbf{k} -algebra homomorphism $\iota : U\mathfrak{F}_{N+1} \rightarrow S$ the image $\iota(\varphi) \in S$ is denoted by $\varphi(\iota(A), \iota(B(0)), \dots, \iota(B(N-1)))$. The Lie algebra structure is given by

$$(2.4) \quad \langle (\varphi_1, \psi_1), (\varphi_2, \psi_2) \rangle = (\langle \varphi_1, \varphi_2 \rangle, \langle \psi_1, \psi_2 \rangle)$$

with

$$\langle \varphi_1, \varphi_2 \rangle = [\varphi_1, \varphi_2] + D_{\varphi_2}(\varphi_1) - D_{\varphi_1}(\varphi_2) \text{ and } \langle \psi_1, \psi_2 \rangle = [\psi_1, \psi_2] + \bar{D}_{\psi_2}(\psi_1) - \bar{D}_{\psi_1}(\psi_2).$$

Here \bar{D}_{ψ} means the derivation of $\mathfrak{t}_{3,N}^0$ defined by

$$\bar{D}_{\psi}(A) = [\psi, A], \quad \bar{D}_{\psi}(B(a)) = [\psi - \psi(A, B(a), B(a+1), \dots, B(a-1)), B(a)]$$

for $a \in \mathbf{Z}/N\mathbf{Z}$ and

$$\bar{D}_{\psi}(C) = [\psi(C, B(0), B(N-1), \dots, B(1)), C].$$

The Lie algebra structure is realised by the embedding

$$\mathfrak{grtm}_{(\bar{1},1)}(N, \mathbf{k}) \hookrightarrow \text{Der}(\mathfrak{t}_3^0) \times \text{Der}(\mathfrak{t}_{3,N}^0)$$

sending $(\varphi, \psi) \mapsto (D_{\varphi}, \bar{D}_{\psi})$.

Remark 2.2. It is shown in [E] that the special derivation condition (2.3) for ψ implies the octagon equation (2.2).

Definition 2.3 ([E]). For $N \geq 1$, the group $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})$ is defined to be the set of pairs $(g, h) \in \exp \mathfrak{t}_3^0 \times \exp \mathfrak{t}_{3,N}^0$ satisfying $g \in \text{GRT}_1(\mathbf{k})$, $c_{B(0)}(h) = 0$, the *mixed pentagon equation* in $\exp \mathfrak{t}_{4,N}^0$

$$(2.5) \quad h^{1,2,34} h^{12,3,4} = g^{2,3,4} h^{1,23,4} h^{1,2,3},$$

the *octagon equation* in $\exp \mathfrak{t}_{3,N}^0$

$$(2.6) \quad h(A, B(1), B(2), \dots, B(0))^{-1} h(C, B(1), B(0), \dots, B(2)) \cdot \\ h(C, B(0), B(N-1), \dots, B(1))^{-1} h(A, B(0), B(1), \dots, B(N-1)) = 1$$

with $A + \sum_{a \in \mathbf{Z}/N\mathbf{Z}} B(a) + C = 0$ and

the *special action condition* in $\exp \mathfrak{t}_{3,N}^0$

$$(2.7) \quad A + \sum_{a \in \mathbf{Z}/N\mathbf{Z}} \text{Ad}(\tau_a h^{-1})(B(a)) + \text{Ad}\left(h^{-1} \cdot h(C, B(0), B(N-1), \dots, B(1))\right)(C) = 0$$

² For our convenience, we slightly change the original definition by adding the small condition $c_{B(0)}(\psi) = 0$. The relation to the original Lie algebra is the direct sum decomposition of Lie algebras $\mathfrak{g}^{\text{original}} = \mathfrak{g} \oplus \mathbf{k} \cdot B(0)$, where $\mathfrak{g} = \mathfrak{grtm}_{(\bar{1},1)}(N, \mathbf{k})$.

where τ_a ($a \in \mathbf{Z}/N\mathbf{Z}$) is the automorphism defined by $A \mapsto A$ and $B(c) \mapsto B(c+a)$ for all $c \in \mathbf{Z}/N\mathbf{Z}$.

It forms a group by the multiplication

$$(2.8) \quad (g_1, h_1) \circ (g_2, h_2) = \left(g_2(A, B) \cdot g_1(A, \text{Ad}(g_2^{-1})(B)), h_2(A, B(0), B(1), \dots, B(N-1)) \cdot h_1(A, \text{Ad}(h_2^{-1})B(0), \text{Ad}(\tau_1 h_2^{-1})B(1), \dots, \text{Ad}(\tau_{N-1} h_2^{-1})B(N-1)) \right).$$

The group structure is realised by the embedding (but opposite homomorphism)

$$\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k}) \hookrightarrow \text{Aut} \mathfrak{t}_3^0 \times \text{Aut} \mathfrak{t}_{3,N}^0$$

sending (g, h) to the automorphism (A_g, \bar{A}_h) where \bar{A}_h is defined by $A \mapsto A$ and $B(a) \mapsto \text{Ad}(\tau_a h^{-1})(B(a))$ for $a \in \mathbf{Z}/N\mathbf{Z}$. We note that its associated Lie algebra is $\mathfrak{grtm}_{(\bar{1},1)}(\mathbf{k})$.

Definition 2.4 ([E]). The torsor $\text{Pseudo}_{(\bar{1},1)}(N, \mathbf{k})$ is defined to be the set of pairs $(g, h) \in \exp \mathfrak{t}_3^0 \times \exp \mathfrak{t}_{3,N}^0$ satisfying $g \in \text{M}_1(\mathbf{k})$, $c_{B(0)}(h) = 0$, the mixed pentagon equation (2.5) and the following variant of octagon equation in $\exp \mathfrak{t}_{3,N}^0$

$$(2.9) \quad h(A, B(1), B(2), \dots, B(0))^{-1} \exp\left\{\frac{B(1)}{2}\right\} h(C, B(1), B(0), \dots, B(2)) \exp\left\{\frac{C}{N}\right\} \cdot h(C, B(0), B(N-1), \dots, B(1))^{-1} \exp\left\{\frac{B(0)}{2}\right\} \cdot h(A, B(0), B(1), \dots, B(N-1)) \exp\left\{\frac{A}{N}\right\} = 1.$$

It forms a right $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})$ -torsor by (2.8) with $(g_1, h_1) \in \text{Pseudo}_{(\bar{1},1)}(N, \mathbf{k})$ and $(g_2, h_2) \in \text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})$.

Remark 2.5. In contrast with remark 1.4, it is not known if (2.3) and (2.7) follow from the rest of the equations (cf. [E] remark 7.8).

Remark 2.6. In [F10b] it is shown that the mixed pentagon equation (2.5) implies the double shuffle relation and the regularization relation among multiple L -values.

Let $N, N' \geq 1$ with $N'|N$. Put $d = N/N'$. The morphism $\pi_{NN'} : \mathfrak{t}_{n,N} \rightarrow \mathfrak{t}_{n,N'}$ is defined by

$$t^{1i} \mapsto dt^{1i} \text{ and } t^{ij}(a) \mapsto t^{ij}(\bar{a}) \quad (i \neq j, 2 \leq i, j \leq n, a \in \mathbf{Z}/N\mathbf{Z}),$$

where $\bar{a} \in \mathbf{Z}/N'\mathbf{Z}$ means the image of a under the map $\mathbf{Z}/N\mathbf{Z} \rightarrow \mathbf{Z}/N'\mathbf{Z}$.

The morphism $\delta_{NN'} : \mathfrak{t}_{n,N} \rightarrow \mathfrak{t}_{n,N'}$ is defined by

$$t^{1i} \mapsto t^{1i} \text{ and } t^{ij}(a) \mapsto \begin{cases} t^{ij}(a/d) & \text{if } d|a, \\ t^{ij}(a) \mapsto 0 & \text{if } d \nmid a \end{cases}$$

($i \neq j, 2 \leq i, j \leq n, a \in \mathbf{Z}/N\mathbf{Z}$). For $\psi \in \mathfrak{t}_{3,N}^0$, put $\rho_{NN'}(\psi) = c_{B(0)}(\pi_{NN'}(\psi)) - c_{B(0)}(\psi)$.

The morphism $\pi_{NN'}$ (resp. $\delta_{NN'}$) : $\mathfrak{t}_{n,N} \rightarrow \mathfrak{t}_{n,N'}$ induces the morphisms $\mathfrak{grtm}_{(\bar{1},1)}(N, \mathbf{k}) \rightarrow \mathfrak{grtm}_{(\bar{1},1)}(N', \mathbf{k})$, $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k}) \rightarrow \text{GRTM}_{(\bar{1},1)}(N', \mathbf{k})$ and $\text{Pseudo}_{(\bar{1},1)}(N, \mathbf{k}) \rightarrow \text{Pseudo}_{(\bar{1},1)}(N', \mathbf{k})$ which we denote by the same symbol $\pi_{NN'}$ (resp. $\delta_{NN'}$). We also remark that

$$\mathfrak{grtm}_{(\bar{1},1)}(1, \mathbf{k}) = \mathfrak{grt}_1(\mathbf{k}), \text{GRTM}_{(\bar{1},1)}(1, \mathbf{k}) = \text{GRT}_1(\mathbf{k}) \text{ and } \text{Pseudo}_{(\bar{1},1)}(1, \mathbf{k}) = \text{M}_1(\mathbf{k}).$$

Definition 2.7 ([E]). (1). For $N \geq 1$, $\mathfrak{grtm}\mathfrak{d}_{(\bar{1},1)}(N, \mathbf{k})$ is the Lie subalgebra of $\mathfrak{grtm}_{(\bar{1},1)}(N, \mathbf{k})$ defined by imposing the *distribution relation* in $\mathfrak{t}_{3,N'}^0$ for all $N'|N$

$$(2.10) \quad (\pi_{NN'} - \delta_{NN'})(\psi) = \rho_{NN'}(\psi)B(0).$$

(2). For $N \geq 1$, $\text{GRTMD}_{(\bar{1},1)}(N, \mathbf{k})$ is the subgroup of $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})$ defined by imposing the *distribution relation* in $\exp \mathfrak{t}_{3,N'}^0$ for all $N'|N$

$$(2.11) \quad \pi_{NN'}(h) = e^{\rho_{NN'}(h)B(0)}\delta_{NN'}(h).$$

(3). For $N \geq 1$, the $\text{GRTMD}_{(\bar{1},1)}(N, \mathbf{k})$ -torsor $\text{Psdist}_{(\bar{1},1)}(N, \mathbf{k})$ is the subtoror of $\text{Pseudo}_{(\bar{1},1)}(N, \mathbf{k})$ defined by imposing the distribution relation (2.11) in $\exp \mathfrak{t}_{3,N'}^0$ for all $N'|N$.

Remark 2.8. A typical example of an element of $\text{Psdist}_{(\bar{1},1)}(N, \mathbf{C})$ is

$$\varphi_{KZ}^N(A, B(0), \dots, B(N-1)) = \Phi_{KZ}^N\left(\frac{A}{2\pi\sqrt{-1}}, \frac{B(0)}{2\pi\sqrt{-1}}, \dots, \frac{B(N-1)}{2\pi\sqrt{-1}}\right)$$

where $\Phi_{KZ}^N(A, B(0), \dots, B(N-1))$ is the *cyclotomic Drinfeld associator* constructed in [E]. It has the following expression:

$$\begin{aligned} \Phi_{KZ}^N = 1 + \sum (-1)^m L(k_1, \dots, k_m; \zeta_1, \dots, \zeta_m) A^{k_m-1} B(a_m) \dots A^{k_1-1} B(a_1) \\ + (\text{regularized terms}) \end{aligned}$$

where $\zeta_1 = \zeta_N^{a_2-a_1}, \dots, \zeta_{m-1} = \zeta_N^{a_m-a_{m-1}}, \zeta_m = \zeta_N^{-a_m}$ with $\zeta_N = \exp\{\frac{2\pi\sqrt{-1}}{N}\}$ and $L(k_1, \dots, k_m; \zeta_1, \dots, \zeta_m)$ are *multiple L-values* defined by the following series

$$(2.12) \quad L(k_1, \dots, k_m; \zeta_1, \dots, \zeta_m) := \sum_{0 < n_1 < \dots < n_m} \frac{\zeta_1^{n_1} \dots \zeta_m^{n_m}}{n_1^{k_1} \dots n_m^{k_m}}$$

for $m, k_1, \dots, k_m \in \mathbf{N}(= \mathbf{Z}_{>0})$ and $\zeta_1, \dots, \zeta_m \in \mu_N$ with $(k_m, \zeta_m) \neq (1, 1)$.

The following says that the distribution relation for $N' = 1$ follows from the mixed pentagon equation. Note that the distribution relation for $N' = N$ is automatically satisfied.

Proposition 2.9. (1). Suppose that $(\varphi, \psi) \in \mathfrak{t}_3^0 \times \mathfrak{t}_{3,N}^0$ satisfies the mixed pentagon equation (2.1) in $\mathfrak{t}_{4,N}^0$. Then it also satisfies the distribution relation (2.10) for $N' = 1$ in $\mathfrak{t}_{3,1}^0 = \mathfrak{t}_3^0$.

(2). Suppose that $(g, h) \in \exp \mathfrak{t}_3^0 \times \exp \mathfrak{t}_{3,N}^0$ satisfies the mixed pentagon equation (2.5) in $\exp \mathfrak{t}_{4,N}^0$. Then it also satisfies the distribution relation (2.11) for $N' = 1$ in $\exp \mathfrak{t}_{3,1}^0 = \exp \mathfrak{t}_3^0$.

Proof. (1). By taking the image of (2.1) by the composition of π_{N1} with the projection $\mathfrak{t}_{4,1}^0 = \mathfrak{t}_4^0 \rightarrow \mathfrak{t}_3^0$ eliminating the first strand, we get

$$\pi_{N1}(\psi) = \varphi + Nc_A(\psi)A + c_B(\pi_{N1}(\psi))B.$$

Next by taking the image of (2.1) by the composition of δ_{N1} with the projection, we get

$$\delta_{N1}(\psi) = \varphi + c_A(\psi)A + c_{B(0)}(\psi)B.$$

By the lemma below these two equations give (2.10) for $N' = 1$.

(2). Similarly we obtain $\pi_{N1}(h) = e^{c_B(\pi_{N1}(h))B}g$ and $\delta_{N1}(h) = e^{c_{B(0)}(h)B}g$ from (2.5), which implies the claim. \square

Lemma 2.10. *Suppose that $(\varphi, \psi) \in \mathfrak{t}_3^0 \times \mathfrak{t}_{3,N}^0$ (resp. $\in \exp \mathfrak{t}_3^0 \times \exp \mathfrak{t}_{3,N}^0$) satisfies the mixed pentagon equation (2.1) in $\mathfrak{t}_{4,N}^0$ (resp. (2.5) in $\exp \mathfrak{t}_{4,N}^0$). Then $c_A(\psi) = 0$.*

Proof. It can be proved directly by inspecting the terms of degree 1. \square

As a corollary, we have

Corollary 2.11. *For a prime p , we have*

$$\begin{aligned} \mathbf{grtm}\mathfrak{d}_{(\bar{1},1)}(p, \mathbf{k}) &= \mathbf{grtm}_{(\bar{1},1)}(p, \mathbf{k}), \\ \mathbf{GRTMD}_{(\bar{1},1)}(p, \mathbf{k}) &= \mathbf{GRTM}_{(\bar{1},1)}(p, \mathbf{k}), \\ \mathbf{Psdist}_{(\bar{1},1)}(p, \mathbf{k}) &= \mathbf{Pseudo}_{(\bar{1},1)}(p, \mathbf{k}). \end{aligned}$$

Remark 2.12. In [DeG] Deligne and Goncharov construct the motivic fundamental group $\pi_1^M(\mathbf{P}^1 \setminus \{0, 1, \mu_N\}, 1_0)$ (μ_N : the group of N -th roots of unity) with the tangential base point 1_0 at 0, which determines a pro-object of the \mathbf{Q} -linear category $\mathbf{MT}(\mathbf{Z}[\mu_N, \frac{1}{N}])_{\mathbf{Q}}$ of mixed Tate motives of $\mathbf{Z}[\mu_N, \frac{1}{N}]$. This causes the morphism

$$\varphi_N : \mathbf{LieGal}^M(\mathbf{Z}[\mu_N, \frac{1}{N}]) \rightarrow \mathbf{Dert}_{3,N}^0,$$

where $\mathbf{LieGal}^M(\mathbf{Z}[\mu_N, \frac{1}{N}])$ is the motivic Lie algebra of the category. It is a graded free Lie algebra with $\mathrm{rk} K_{2n-1}(\mathbf{Z}[\mu_N, \frac{1}{N}])$ generators in each degree $n > 0$. The map φ_N is shown to be injective for $N = 1$ in [Bw] and for $N = 2, 3, 4$ and 8 in [De]. For $N = 6$, a certain modification of the map φ_N is shown to be injective in [De]. Partial injectivity results for $N = 2p$ (p : a prime) were obtained in [DW]. Because all the defining equations of $\mathbf{grtm}\mathfrak{d}_{(\bar{1},1)}(N, \mathbf{k})$ are geometric, it can be shown that $\mathrm{Im} \varphi_N$ is embedded in $\mathbf{grtm}\mathfrak{d}_{(\bar{1},1)}(N, \mathbf{k}) \subset \mathbf{Dert}_{3,N}^0$. It is one of the fundamental questions to ask if they are equal or not.

3. MIXED PENTAGON AND OCTAGON EQUATIONS

In this section, we focus on the case $N = 2$ and prove that the mixed pentagon equation implies the octagon equation.

Theorem 3.1. *Let $(\varphi, \psi) \in \mathfrak{t}_3^0 \times \mathfrak{t}_{3,2}^0$ be a pair satisfying $c_{B(0)}(\psi) = c_{AB(0)}(\psi) = 0$ and the mixed pentagon equation (2.1) in $\mathfrak{t}_{4,2}^0$, i.e.*

$$\begin{aligned} &\psi(t^{12}, t_+^{23} + t_+^{24}, t_-^{23} + t_-^{24}) + \psi(t^{13} + t_+^{23} + t_-^{23}, t_+^{34}, t_-^{34}) \\ &= \varphi(t_+^{23}, t_+^{34}) + \psi(t^{12} + t^{13} + t_+^{23} + t_-^{23}, t_+^{24} + t_+^{34}, t_-^{24} + t_-^{34}) + \psi(t^{12}, t_+^{23}, t_-^{23}) \end{aligned}$$

where $t_+^{ij} = t^{ij}(0)$ and $t_-^{ij} = t^{ij}(1)$. Then ψ satisfies the octagon equation (2.2).

Proof. By taking the image of (2.1) by δ_{21} and eliminating the first strand we get $\delta_{21}(\psi) = \varphi + c_A(\psi)A$, which by lemma 2.10 implies $\delta_{21}(\psi) = \varphi$. Then applying again δ_{21} to the mixed pentagon equation (2.1), we get the equation (1.3) for φ . Then by our assumption $c_B(\varphi) = c_{AB}(\varphi) = 0$ and Theorem 1.6 (1), we have (1.1) for φ .

For $(\varphi, \psi) \in \mathfrak{t}_3^0 \times \mathfrak{t}_{3,2}^0$, put

$$\Pi = \varphi^{2,3,4} + \psi^{1,23,4} + \psi^{1,2,3} - \psi^{1,2,34} - \psi^{12,3,4}$$

in $\mathfrak{t}_{4,2}^0$. Let \mathfrak{S}_3 be the group of permutations of $\{1, 2, 3, 4\}$ which fix $\{1\}$. Then

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \Pi^{1,\sigma(2),\sigma(3),\sigma(4)} &= (\psi^{1,2,3} - \psi^{1,3,2}) + (\psi^{14,3,2} - \psi^{14,2,3}) + (\psi^{13,2,4} - \psi^{13,4,2}) \\ &\quad + (\psi^{12,4,3} - \psi^{12,3,4}) + (\psi^{1,3,4} - \psi^{1,4,3}) + (\psi^{1,4,2} - \psi^{1,2,4}) + \sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \varphi^{\sigma(2),\sigma(3),\sigma(4)}. \end{aligned}$$

There is a unique automorphism s of the Lie algebra $\mathfrak{t}_{4,2}^0$ such that

$$s(t^{12}) = t^{13}, s(t^{13}) = t^{12}, s(t^{14}) = t^{14}, s(t_{\pm}^{23}) = t_{\mp}^{23}, s(t_{\pm}^{24}) = t_{\mp}^{34} \text{ and } s(t_{\pm}^{34}) = t_{\pm}^{24}.$$

Then $s^4 = id$ and

$$s(\psi^{12,3,4}) = \psi^{13,2,4}, s(\psi^{12,4,3}) = \psi^{13,4,2}, s(\psi^{1,4,3}) = \psi^{1,4,2}, s(\psi^{1,3,4}) = \psi^{1,2,4}.$$

It follows that

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \Pi^{1,\sigma(2),\sigma(3),\sigma(4)} &= (\psi^{1,2,3} - \psi^{1,3,2}) + (\psi^{14,3,2} - \psi^{14,2,3}) \\ &\quad + (s - id)(\psi^{12,3,4} - \psi^{12,4,3} + \psi^{1,4,3} - \psi^{1,3,4}) + \sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \varphi^{\sigma(2),\sigma(3),\sigma(4)}. \end{aligned}$$

Hence

$$\begin{aligned} (id + s + s^2 + s^3) \sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \Pi^{1,\sigma(2),\sigma(3),\sigma(4)} &= (id + s + s^2 + s^3)(\psi^{1,2,3} - \psi^{1,3,2} \\ &\quad + \psi^{14,3,2} - \psi^{14,2,3}) + (id + s + s^2 + s^3) \sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \varphi^{\sigma(2),\sigma(3),\sigma(4)}. \end{aligned}$$

By (2.1), $\Pi = 0$. By (1.1), the last term is 0. So we have

$$(id + s + s^2 + s^3)(\psi^{1,2,3} - \psi^{1,3,2} + \psi^{14,3,2} - \psi^{14,2,3}) = 0.$$

Since $s^2(X) = X$ for $X = \psi^{1,2,3}, \psi^{1,3,2}, \psi^{14,3,2}$ and $\psi^{14,2,3}$,

$$(id + s)(\psi^{1,2,3} - \psi^{1,3,2} + \psi^{14,3,2} - \psi^{14,2,3}) = 0.$$

Let s' be the automorphism of $\mathfrak{t}_{3,2}^0$ uniquely defined by s sending

$$s(t^{12}) = t^{13}, s(t^{13}) = t^{12} \text{ and } s(t_{\pm}^{23}) = t_{\mp}^{23}.$$

Then the above equation can be read as

$$\Omega^{1,2,3} = \Omega^{14,2,3} \quad \text{in } \mathfrak{t}_{4,2}^0$$

where

$$\Omega := \psi^{1,2,3} - \psi^{1,3,2} + s'(\psi)^{1,2,3} - s'(\psi)^{1,3,2} \in \mathfrak{t}_{3,2}^0.$$

By the lemma below, Ω is described as $\Omega = r(t_{+}^{23}, t_{-}^{23})$ for $r \in \mathfrak{F}_2$. So

$$\psi(t^{12}, t_{+}^{23}, t_{-}^{23}) - \psi(t^{13}, t_{+}^{23}, t_{-}^{23}) + \psi(t^{13}, t_{-}^{23}, t_{+}^{23}) - \psi(t^{12}, t_{-}^{23}, t_{+}^{23}) = r(t_{+}^{23}, t_{-}^{23}).$$

By the identifications $\mathfrak{t}_{3,2}^0/(t^{12}) \simeq \mathfrak{F}_2 \simeq \mathfrak{t}_{3,2}^0/(t^{13})$, we have

$$\psi(0, t_{+}^{23}, t_{-}^{23}) - \psi(-t_{+}^{23} - t_{-}^{23}, t_{+}^{23}, t_{-}^{23}) + \psi(-t_{+}^{23} - t_{-}^{23}, t_{-}^{23}, t_{+}^{23}) - \psi(0, t_{-}^{23}, t_{+}^{23}) = r(t_{+}^{23}, t_{-}^{23}),$$

$$\psi(-t_{+}^{23} - t_{-}^{23}, t_{+}^{23}, t_{-}^{23}) - \psi(0, t_{+}^{23}, t_{-}^{23}) + \psi(0, t_{-}^{23}, t_{+}^{23}) - \psi(-t_{+}^{23} - t_{-}^{23}, t_{-}^{23}, t_{+}^{23}) = r(t_{+}^{23}, t_{-}^{23}).$$

These equalities give $r = 0$, which means $\Omega = 0$. It yields the validity of the octagon equation (2.2) for ψ . \square

Lemma 3.2. *If $X \in \mathfrak{t}_{3,2}^0$ satisfies $X^{1,2,3} = X^{14,2,3}$ in $\mathfrak{t}_{4,2}^0$, then X belongs to the free Lie subalgebra \mathfrak{F}_2 of rank 2 with generators t_+^{23} and t_-^{23} .*

Proof. Consider the linear map $F : \mathfrak{t}_{3,2}^0 \rightarrow \mathfrak{t}_{4,2}^0$ sending $h \mapsto h^{1,2,3} - h^{14,2,3}$. Its image is contained in the Lie subalgebra of $\mathfrak{t}_{4,2}^0$ generated by t^{12} , t_{\pm}^{23} , t_{\pm}^{24} . According to [E], this Lie algebra is freely generated by these 5 elements. On the other hand, $\mathfrak{t}_{3,2}^0$ can be identified with the free Lie algebra \mathfrak{F}_3 generated by t^{12} , t_{\pm}^{23} . It follows that $\ker F$ is equal to the kernel of the map $\tilde{F} : \mathfrak{F}_3 \rightarrow \mathfrak{F}_5$ sending

$$h \mapsto h(t^{12}, t_+^{23}, t_-^{23}) - h(t^{12} + t_+^{24} + t_-^{24}, t_+^{23}, t_-^{23}).$$

If $X \in \ker \tilde{F}$, then $X(0, t_+^{23}, t_-^{23}) = X(t_+^{24}, t_+^{23}, t_-^{23})$, which implies, as the Lie subalgebra of \mathfrak{F}_5 generated by t_+^{24} , t_+^{23} and t_-^{23} is isomorphic to \mathfrak{F}_3 , that X belongs to the Lie subalgebra $\mathfrak{F}_2 \subset \mathfrak{F}_3$ freely generated by t_+^{23} and t_-^{23} . \square

The following is a geometric interpretation of our arguments above.

Remark 3.3. Put $\tilde{\mathfrak{M}}_{0,4}^2 := \{z \in \mathbf{A}^1 | z \neq 0, \pm 1\}$ and

$$\tilde{\mathfrak{M}}_{0,5}^2 := \{(x, y) \in \mathbf{A}^2 | xy \neq \pm 1, x, y \neq 0, \pm 1\}.$$

These are the Kummer coverings of the moduli spaces

$\mathfrak{M}_{0,4} := \{z \in \mathbf{A}^1 | z \neq 0, 1\}$ with $\tilde{\mathfrak{M}}_{0,4}^2 \rightarrow \mathfrak{M}_{0,4} : z \mapsto z^2$ and

$\mathfrak{M}_{0,5} := \{(x, y) \in \mathbf{A}^2 | xy \neq 1, x, y \neq 0, 1\}$ with $\tilde{\mathfrak{M}}_{0,5}^2 \rightarrow \mathfrak{M}_{0,5} : (x, y) \mapsto (x^2, y^2)$

respectively. The Lie algebras \mathfrak{t}_3^0 , $\mathfrak{t}_{3,2}^0$, \mathfrak{t}_4^0 and $\mathfrak{t}_{4,2}^0$ are associated with the fundamental groups of $\mathfrak{M}_{0,4}$, $\tilde{\mathfrak{M}}_{0,4}^2$, $\mathfrak{M}_{0,5}$ and $\tilde{\mathfrak{M}}_{0,5}^2$ respectively. The picture of $\tilde{\mathfrak{M}}_{0,5}^2$ in Figure 1 is obtained by blowing-ups of \mathbf{A}^2 at $(x, y) = (0, 0)$, $(\pm 1, \pm 1)$ and (∞, ∞) . Our Π above corresponds to the pentagon near origin surrounded by $\varphi^{2,3,4}$, $\psi^{1,23,4}$, $\psi^{1,2,3}$, $\psi^{1,2,34}$ and $\psi^{12,3,4}$. Our $\sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \Pi^{1,\sigma(2),\sigma(3),\sigma(4)}$ above corresponds to the six pentagons in the first quadrant and $\sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \varphi^{\sigma(2),\sigma(3),\sigma(4)}$ corresponds to the hexagon there. Our $(id + s + s^2 + s^3) \sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \Pi^{1,\sigma(2),\sigma(3),\sigma(4)}$ stands for all the (24-)pentagons in the picture and $(id + s + s^2 + s^3)(\psi^{1,2,3} - \psi^{1,3,2} + \psi^{14,3,2} - \psi^{14,2,3})$ means the two octagons near $(0, 0)$ and (∞, ∞) .

Next we show an analogue of Theorem 3.1 for group-like series.

Theorem 3.4. *Let $(g, h) \in \exp \mathfrak{t}_3^0 \times \exp \mathfrak{t}_{3,2}^0$ be a pair satisfying $c_{B(0)}(h) = c_{AB(0)}(h) = 0$, the mixed pentagon equation (2.5) and the special action condition (2.7). Then h satisfies the octagon equation (2.6).*

Proof. By taking the image of (2.5) by δ_{21} and eliminating the first strand, we get $\delta_{21}(h) = g$ because of Lemma 2.10 and then the pentagon equation (1.7) for g . By $c_{B(0)}(h) = 0$ and (1.7), the linear terms of g are all zero. Hence by $c_{AB(0)}(h) = 0$, its quadratic terms are all zero. By Theorem 1.6.(2), $g \in \text{GRT}_1(\mathbf{k})$. So it suffices to prove $(g, h) \in \text{GRTM}_{(\bar{1},1)}(2, \mathbf{k})$. The proof can be done by induction on degree. Suppose that we have (2.6) for (g, h) modulo degree n , which we denote as

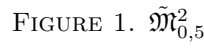
$$(g, h) \pmod{\deg n} \in \text{GRTM}_{(\bar{1},1)}(2, \mathbf{k})^{(n)}.$$

Then there is a pair

$$(g_1, h_1) \in \text{GRTM}_{(\bar{1},1)}(2, \mathbf{k}) \text{ with } (g, h) \equiv (g_1, h_1) \pmod{\deg n}$$

by Lemma A.3. Let (g_0, h_0) be the pair defined by

$$(g, h) = (g_0, h_0) \circ (g_1, h_1).$$


$$(\varphi, \psi) \in \mathfrak{grtm}_{(\bar{1}, 1)}(2, \mathbf{k}).$$
$$(g_0, h_0) \pmod{\deg n + 1} \in \text{GRTM}_{(\bar{1}, 1)}(2, \mathbf{k})^{(n+1)}.$$
☐

Remark 3.5. We note that in Theorem 3.1 we do not assume the special condition (2.3), on the other hand, in Theorem 3.4 we assume the special condition (2.7). The analogue of Theorem 1.6 (3) might be the implication of (2.9) from (2.1) but we do not know whether this implication holds.

4. BROADHURST DUALITY

We will show that the Broadhurst duality relation is compatible with the torsor structure of $\text{Pseudo}_{(\bar{1},1)}(2, \mathbf{k})$.

Let τ be the involution of $\mathfrak{t}_{3,2}^0$ defined by $\tau : A \leftrightarrow B(0)$ and $B(1) \leftrightarrow C$.

Definition 4.1. (1). The set $\mathbf{grtm}_{(\bar{1},1)}(2, \mathbf{k})$ is defined as the set of all $\psi \in \mathbf{grtm}_{(\bar{1},1)}(2, \mathbf{k})$ such that the *Broadhurst duality relation*

$$(4.1) \quad \tau(\psi) + \psi + \alpha_\psi(A + B(0)) = 0$$

holds for some $\alpha_\psi \in \mathbf{k}$.

(2). The set $\text{GRTM}_{(\bar{1},1)}(2, \mathbf{k})$ is defined as the set of all $(g, h) \in \text{GRTM}_{(\bar{1},1)}(2, \mathbf{k})$ such that the *Broadhurst duality relation*

$$(4.2) \quad \tau(h)e^{\alpha_h B(0)} h e^{\alpha_h A} = 1$$

holds for some $\alpha_h \in \mathbf{k}$.

(3). The set $\text{PseudoB}_{(\bar{1},1)}(2, \mathbf{k})$ is defined as the set of all $(g, h) \in \text{Pseudo}_{(\bar{1},1)}(2, \mathbf{k})$ such that the *Broadhurst duality relation* (4.2) holds for some $\alpha_h \in \mathbf{k}$.

Actually α_ψ and α_h are equal to the coefficients of $B(1)$ in ψ and h respectively.

Theorem 4.2. (1). The set $\mathbf{grtm}_{(\bar{1},1)}(2, \mathbf{k})$ forms a Lie algebra by the Lie bracket (1.4).

(2). The set $\text{GRTM}_{(\bar{1},1)}(2, \mathbf{k})$ forms an algebraic group by the multiplication (1.8) and its associated Lie algebra is $\mathbf{grtm}_{(\bar{1},1)}(2, \mathbf{k})$.

(3). The set $\text{PseudoB}_{(\bar{1},1)}(2, \mathbf{k})$ forms a right $\text{GRTM}_{(\bar{1},1)}(2, \mathbf{k})$ -torsor by (1.8).

Proof. (1). Put $\text{OutDer}(\mathfrak{t}_{3,2}^0) = (\text{Der}/\text{Int})(\mathfrak{t}_{3,2}^0)$. This quotient forms a Lie algebra with the involution induced by τ . Its invariant part $\text{OutDer}^+(\mathfrak{t}_{3,2}^0)$ again forms a Lie algebra. The embedding sending $(\varphi, \psi) \mapsto (D_\varphi, \bar{D}_\psi)$ induces the embedding $\mathbf{grtm}_{(\bar{1},1)}(2, \mathbf{k}) \hookrightarrow \text{Der}(\mathfrak{t}_3^0) \times \text{OutDer}(\mathfrak{t}_{3,2}^0)$. It can be checked that (4.1) is the condition for (φ, ψ) to belong to the intersection of two Lie algebras $\mathbf{grtm}_{(\bar{1},1)}(2, \mathbf{k})$ and $\text{Der}(\mathfrak{t}_3^0) \times \text{OutDer}^+(\mathfrak{t}_{3,2}^0)$.

(2). It can be proved similarly. Put $\text{Out}\mathfrak{t}_{3,2}^0 = (\text{Aut}/\text{Inn})(\mathfrak{t}_{3,2}^0)$, the outer automorphism group of $\mathfrak{t}_{3,2}^0$; the group of automorphisms modulo inner automorphisms. This quotient forms a group with the involution induced from τ . Its invariant part $\text{Out}^+\mathfrak{t}_{3,2}^0$ again forms a group. The embedding sending $(g, h) \mapsto (A_g, \bar{A}_h)$ induces the embedding $\text{GRTM}_{(\bar{1},1)}(2, \mathbf{k}) \hookrightarrow \text{Aut}(\mathfrak{t}_3^0) \times \text{Out}(\mathfrak{t}_{3,2}^0)$. It can be checked that (4.2) is the condition for (g, h) to belong to the intersection of two group $\text{GRTM}_{(\bar{1},1)}(2, \mathbf{k})$ and $\text{Aut}(\mathfrak{t}_3^0) \times \text{Out}^+(\mathfrak{t}_{3,2}^0)$. Since $\mathbf{grtm}_{(\bar{1},1)}(2, \mathbf{k})$ and $\text{Der}(\mathfrak{t}_3^0) \times \text{OutDer}^+(\mathfrak{t}_{3,2}^0)$ are the associated Lie algebras with these two groups, $\mathbf{grtm}_{(\bar{1},1)}(2, \mathbf{k})$ is associated with $\text{GRTM}_{(\bar{1},1)}(2, \mathbf{k})$.

(3). By direct calculation, it can be shown that

$$\tau(h_3)e^{(\alpha_{h_1} + \alpha_{h_2})B(0)} h_3 e^{(\alpha_{h_1} + \alpha_{h_2})A} = 1$$

for $(g_3, h_3) = (g_1, h_1) \circ (g_2, h_2)$ with $(g_1, h_1) \in \text{PseudoB}_{(\bar{1},1)}(2, \mathbf{k})$ and $(g_2, h_2) \in \text{GRTMB}_{(\bar{1},1)}(2, \mathbf{k})$, which shows that $\text{PseudoB}_{(\bar{1},1)}(2, \mathbf{k})$ is a $\text{GRTMB}_{(\bar{1},1)}(2, \mathbf{k})$ -space. To prove that it forms a torsor, it suffices to show that the action is transitive. Assume that (g_1, h_1) and (g_3, h_3) belong to $\text{PseudoB}_{(\bar{1},1)}(2, \mathbf{k})$ and they are equal mod $\deg n - 1$. Then the degree n -part ψ of their difference satisfies $\tau(\psi) + \psi = 0$. So it gives an element $(\varphi, \psi) \in \text{grtmdb}_{(\bar{1},1)}(2, \mathbf{k})$. Put

$$(g_2^{(n)}, h_2^{(n)}) := \text{Exp}(\varphi, \psi) \in \text{GRTMB}_{(\bar{1},1)}(2, \mathbf{k}).$$

Let $(g_2, h_2) \in \text{GRTM}_{(\bar{1},1)}(2, \mathbf{k})$ be the element uniquely determined by $(g_3, h_3) = (g_1, h_1) \circ (g_2, h_2)$. Then $(g_2, h_2) \equiv (g_2^{(n)}, h_2^{(n)}) \pmod{\deg n}$. By approximation methods replacing (g_1, h_1) by $(g_1, h_1) \circ (g_2^{(n)}, h_2^{(n)})$, we can show (g_2, h_2) belongs to $\text{GRTMB}_{(\bar{1},1)}(2, \mathbf{k})$. \square

We note that by Corollary 2.11

$$\begin{aligned} \text{grtmdb}_{(\bar{1},1)}(2, \mathbf{k}) &= \text{grtmdb}_{(\bar{1},1)}(2, \mathbf{k}), \\ \text{GRTMDB}_{(\bar{1},1)}(2, \mathbf{k}) &= \text{GRTMB}_{(\bar{1},1)}(2, \mathbf{k}), \\ \text{PsdistB}_{(\bar{1},1)}(2, \mathbf{k}) &= \text{PseudoB}_{(\bar{1},1)}(2, \mathbf{k}). \end{aligned}$$

Remark 4.3. (i). The equation (4.2) holds for $h = \Phi_{KZ}^N$ with $\alpha = \log 2$ and $N = 2$ (cf. [O] §4). Here Φ_{KZ}^N means an N -cyclotomic analogue of the Drinfeld associator, whose all coefficients are multiple L -values (see also [E]). It is explained in [O] that the equation yields the Broadhurst duality relation [Bh] (127) of multiple L -values with signature $\{\pm\}$.

(ii). In [LNS], a subgroup \mathbb{I} of the pro-finite Grothendieck-Teichmüller group \widehat{GT} is introduced, with the properties of both containing the absolute Galois group $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ of the rational number field \mathbf{Q} and of acting on the pro-finite completion of all the mapping class groups. One of its main defining conditions is Equation (IV) from [LNS], an equivalent form of which was found in [F03a] Equation (3). The latter equation is a pro-finite analogue of (4.2).

Since (4.1) is geometric, $\text{grtmdb}_{(\bar{1},1)}(2, \mathbf{k})$ contains the free Lie algebra $\text{LieGal}^M(\mathbf{Z}[\frac{1}{2}])$ with one free generator in each degree 1, 3, 5, 7, ... It is fundamental to ask if they are equal or not. Namely

Question 4.4. Is the Lie algebra $\text{grtmdb}_{(\bar{1},1)}(2, \mathbf{k})$ free with one generator in each degree 1, 3, 5, 7, ...?

APPENDIX A. INFINITESIMAL MODULE CATEGORIES

In this appendix basics of infinitesimal module categories are given. We prove the fact implicitly employed in [E] that $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})$ forms a group by using its action on infinitesimal module categories.

A.1. Infinitesimal braided monoidal categories. An *infinitesimal braided monoidal category* (IBMC for short) is a set

$$\mathbb{C} = (\mathcal{C}, \otimes, I, a, c, l, r, U, t)$$

consisting of a category \mathcal{C} , a bi-functor $\otimes : \mathcal{C}^2 \rightarrow \mathcal{C}$, $I \in \text{Ob}\mathcal{C}$, functorial assignments $a_{XYZ} \in \text{Isom}_{\mathcal{C}}(X \otimes (Y \otimes Z), (X \otimes Y) \otimes Z)$ and $c_{XY} \in \text{Isom}_{\mathcal{C}}(X \otimes Y, Y \otimes X)$,

$l_X \in \text{Isom}_{\mathcal{C}}(I \otimes X, X)$, $r_X \in \text{Isom}_{\mathcal{C}}(X \otimes I, X)$, a normal subgroup U_X of $\text{Aut}_{\mathcal{C}} X$ and $t_{XY} \in \text{Lie} U_{X \otimes Y}$ for all $X, Y, Z \in \text{Ob} \mathcal{C}$ which satisfies the following:

(i). It forms a *braided monoidal category* [M] (quasi-tensor category [Dr]): *the triangle, the pentagon and the hexagon axioms* hold for a, c, l, r and I (cf. [Dr] (1.7)-(1.9b)).

(ii). The group U_X is a pro-unipotent \mathbf{k} -algebraic group and $fU_X f^{-1} = U_Y$ for any $f \in \text{Isom}_{\mathcal{C}}(X, Y)$ holds and any $X, Y \in \text{Ob} \mathcal{C}$.

(iii). The map t_{XY} is functorial on X and Y and satisfies

$$t_{X \otimes Y, Z} = a_{XYZ}(id_X \otimes t_{YZ})a_{XYZ}^{-1} + (c_{YX} \otimes id_Z)a_{YXZ}(id_Y \otimes t_{XZ})a_{YXZ}^{-1}(c_{YX} \otimes id_Z)^{-1}$$

and $c_{XY}t_{XY} = t_{YX}c_{XY}$.

For a group G and a (braided) monoidal category \mathcal{C} , an *action of G on \mathcal{C}* is a collection of morphisms $G \rightarrow \text{Aut}_{\mathcal{C}} X : g \mapsto g_X$ for $X \in \text{Ob} \mathcal{C}$ such that $fg_X = g_Y f$ for any $f \in \text{Isom}_{\mathcal{C}}(X, Y)$ and $g_{X \otimes Y} = g_X \otimes g_Y$ for any $g \in G$ and $X, Y \in \text{Ob} \mathcal{C}$. Let C_N be the cyclic group \mathbf{Z}/N of order $N \in \mathbf{N}$ with a generator σ . We call an IBMC with C_N -action a C_N -IBMC.

For $n \geq 1$, let $\mathfrak{u}_{n,N}$ denote be the completed \mathbf{k} -Lie algebra with generators

$$t(a)^{ij} \quad (i \neq j, 1 \leq i, j \leq n, a \in C_N)$$

and relations

$$t(a)^{ij} = t(-a)^{ji}, [t(a)^{ij}, t(a+b)^{ik} + t(b)^{jk}] = 0 \text{ and } [t(a)^{ij}, t(b)^{kl}] = 0$$

for all $a, b \in C_N$ and all distinct i, j, k, l ($1 \leq i, j, k, l \leq n$). Denote by $G_{n,N}$ the semi-direct product of C_N^n by the symmetric group S_n . This group acts on $\mathfrak{u}_{n,N}$ by

$$(c_1, \dots, c_n) \cdot t(a)^{ij} = t(a + c_i - c_j) \text{ and } s \cdot t(a)^{ij} = t(a)^{s(i), s(j)}$$

for $1 \leq i, j \leq n$, $a, c_1, \dots, c_n \in C_N$ and $s \in S_n$. Put $\mathcal{U}_{n,N} = \exp \mathfrak{u}_{n,N}$ and $\mathcal{G}_{n,N} = \mathcal{U}_{n,N} \rtimes G_{n,N}$. Let \mathbb{C}_{univ} be the category defined by

$$\text{Ob} \mathbb{C}_{\text{univ}} = \coprod_{n \geq 0} \{\text{parenthesizations of the word } \underbrace{\bullet \cdots \bullet}_n\}$$

and for $X, X' \in \text{Ob} \mathbb{C}_{\text{univ}}$,

$$\text{Mor}_{\mathbb{C}_{\text{univ}}}(X, X') = \begin{cases} \mathcal{G}_{n,N} & \text{if their lengths } |X| \text{ and } |X'| \text{ are equal to } n, \\ \emptyset & \text{if their lengths are different.} \end{cases}$$

Let $\otimes : (\text{Ob} \mathbb{C}_{\text{univ}})^2 \rightarrow \text{Ob} \mathbb{C}_{\text{univ}}$ be the map induced by the concatenation and $\mathcal{G}_{m,N} \times \mathcal{G}_{n,N} \rightarrow \mathcal{G}_{m+n,N}$ be the homomorphism induced by the juxtaposition. They yield a morphism $\text{Mor}(X, X') \times \text{Mor}(Y, Y') \rightarrow \text{Mor}(X \otimes Y, X' \otimes Y')$. Put $a_{XYZ} := 1 \in \mathcal{G}_{|X|+|Y|+|Z|,N}$ and

$$c_{XY} := \sigma_{|X|,|Y|} \in S_{|X|+|Y|} \subset G_{|X|+|Y|,N} \subset \mathcal{G}_{|X|+|Y|,N}$$

where $\sigma_{|X|,|Y|}$ means the permutation interchanging X and Y . Set $I = \emptyset$ (the empty word) $\in \text{Ob} \mathbb{C}_{\text{univ}}$, l_X and r_X to be the identity maps. Finally we put $U_X = \mathcal{U}_{m,N} \in \text{Aut}_{\mathbb{C}_{\text{univ}}}(X)$ and

$$t_{XY} := \sum_{1 \leq i \leq m, 1 \leq j \leq n, a \in C_N} t(a)^{i,m+j} \in \mathfrak{u}_{m+n,N}$$

for $|X| = m$ and $|Y| = n$. Then \mathbb{C}_{univ} forms a C_N -IBMC, which is universal in the following sense: if \mathbb{C} is a C_N -IBMC with a distinguished object X , then there exists a unique functor $\mathbb{C}_{\text{univ}} \rightarrow \mathbb{C}$ of C_N -IBMC's which sends \bullet to X .

A.2. Infinitesimal module categories over C_N -braided monoidal categories.

Let $\mathbb{C} = (\mathcal{C}, \otimes, I, a, c, l, r, U, t, \sigma)$ be a C_N -IBMC. An *infinitesimal (right) module category* (IMC for short) over \mathbb{C} is a set

$$\mathbb{M} = (\mathcal{M}, \otimes, b, r, V, t)$$

consisting of a category \mathcal{M} , a bi-functor $\otimes : \mathcal{M} \otimes \mathcal{C} \rightarrow \mathcal{M}$, functorial assignments $b_{MXY} \in \text{Isom}_{\mathcal{M}}(M \otimes (X \otimes Y), (M \otimes X) \otimes Y)$ and $r_M \in \text{Isom}_{\mathcal{M}}(M \otimes I, M)$, a normal subgroup V_M of $\text{Aut}_{\mathcal{M}} M$ and $t_{MX} \in \text{Lie} V_{M \otimes X}$ for all $M \in \text{Ob} \mathcal{M}$ and $X, Y \in \text{Ob} \mathcal{C}$ which satisfies the following:

(I). It forms a *right module category* over $(\mathcal{C}, \otimes, I, a, c, l, r)$: the *mixed pentagon axiom*

$$(b_{MXY} \otimes id_Z) b_{M, X \otimes Y, Z} = b_{M \otimes X, Y, Z} b_{M, X, Y \otimes Z} (id_M \otimes a_{XYZ})$$

and the *triangle axioms*

$$r_{M \otimes X} b_{MXI} = id_M \otimes r_X \quad \text{and} \quad (r_M \otimes id_X) b_{MIX} = id_M \otimes l_X$$

hold for all $M \in \text{Ob} \mathcal{M}$ and $X, Y, Z \in \text{Ob} \mathcal{C}$.

(II). The *octagon axiom*

$$id_{M \otimes X} \otimes \sigma_Y = b_{MXY} (id_M \otimes c_{Y \otimes X}) b_{MYX}^{-1} ((id_M \otimes \sigma_Y) \otimes id_X) b_{MYX}^{-1} \cdot (id_M \otimes c_{XY}) b_{MXY}$$

holds for all $M \in \text{Ob} \mathcal{M}$ and $X, Y \in \text{Ob} \mathcal{C}$.

(III). The group V_M is a pro-unipotent \mathbf{k} -algebraic group and $fV_M f^{-1} = V_{M'}$ holds for any $f \in \text{Isom}_{\mathcal{M}}(M, M')$ and any $M, M' \in \text{Ob} \mathcal{M}$.

(IV). The map t_{MX} is functorial on M and X and satisfies

$$\begin{aligned} t_{M \otimes X, Y} &= b_{MXY} (id_M \otimes c_{YX}) b_{MYX}^{-1} \cdot (t_{MY} \otimes id_X) b_{MYX} (id_M \otimes c_{XY}) b_{MXY}^{-1} \\ &\quad + \sum_{a \in C_N} (id_{M \otimes X} \otimes \sigma_Y^a) \cdot b_{MXY} (id_M \otimes t_{XY}) b_{MXY}^{-1} \cdot (id_{M \otimes X} \otimes \sigma_Y^{-a}) \end{aligned}$$

and

$$t_{M \otimes X, Y} + t_{MX} \otimes id_Y = b_{MXY} t_{M, X \otimes Y} b_{MXY}^{-1}.$$

We can formulate the notion of functors between two IMC's over C_N -IBMC's. We note that such category forms a braided module category in the sense of [E].

A natural morphism $\mathbf{u}_{n,N} \rightarrow \mathbf{t}_{n+1,N}$ is obtained by shifting indices by 1. By the morphism we extend the $G_{n,N}$ -action on $\mathbf{u}_{n,N}$ into on $\mathbf{t}_{n+1,N}$ via

$$(c_1, \dots, c_n) \cdot t^{1,i+1} = t^{1,i+1} \quad \text{and} \quad \sigma(t^{1,i+1}) = t^{1,\sigma(i)+1}$$

for $c_1, \dots, c_n \in C_N$, $\sigma \in S_n$ and $1 \leq i \leq n$. Put $\tilde{\mathcal{U}}_{n+1,N} := \exp \mathbf{t}_{n+1,N}$ and $\tilde{\mathcal{G}}_{n+1,N} := \tilde{\mathcal{U}}_{n+1,N} \rtimes G_{n,N}$. We now construct an IMC \mathbb{M}_{univ} over the C_N -IBMC \mathbb{C}_{univ} . Set

$$\text{Ob} \mathbb{M}_{\text{univ}} = \text{Ob} \mathbb{C}_{\text{univ}}$$

and for $M, M' \in \text{Ob} \mathbb{M}_{\text{univ}}$,

$$\text{Mor}_{\mathbb{M}_{\text{univ}}}(M, M') = \begin{cases} \tilde{\mathcal{G}}_{n+1,N} & \text{if their lengths } |M| \text{ and } |M'| \text{ are equal to } n, \\ \emptyset & \text{if their lengths are different.} \end{cases}$$

Let $\otimes : \text{Ob} \mathbb{M}_{\text{univ}} \times \text{Ob} \mathbb{C}_{\text{univ}} \rightarrow \text{Ob} \mathbb{C}_{\text{univ}}$ be the map induced by the concatenation and $\tilde{\mathcal{G}}_{m+1,N} \times \mathcal{G}_{n,N} \rightarrow \tilde{\mathcal{G}}_{m+n+1,N}$ be the homomorphism induced by the juxtaposition. They yield a morphism $\text{Mor}(M, M') \times \text{Mor}(X, X') \rightarrow \text{Mor}(M \otimes X, M' \otimes X')$.

Put $b_{MXY} := 1 \in \mathcal{G}_{|M|+|X|+|Y|+1,N}$ and $r_M := id_M$. Finally we put $V_M = \tilde{\mathcal{U}}_{m+1,N} \subset \tilde{\mathcal{G}}_{m+1,N} = \text{Aut}_{\mathbb{M}_{\text{univ}}}(M)$ and

$$t_{MX} := \sum_{1 \leq j \leq n} \sum_{0 \leq i \leq m+j-1} t^{i+1,j+m+1} \in \mathfrak{t}_{m+n+1,N}$$

for $|M| = m$ and $|X| = n$. Then it can be shown that \mathbb{M}_{univ} forms an IMC over the C_N -IBMC \mathbb{C}_{univ} , which is universal in the following sense: if \mathbb{M} is an IMC over a C_N -IBMC \mathbb{C} with distinguished objects $M \in \text{Ob}\mathbb{M}$ and $X \in \text{Ob}\mathbb{C}$, then there exists unique functors $\mathbb{M}_{\text{univ}} \rightarrow \mathbb{M}$ and $\mathbb{C}_{\text{univ}} \rightarrow \mathbb{C}$ of IMC's over C_N -IBMC's which send \bullet to M and X respectively.

A.3. Automorphisms. Let $\mathbb{C} = (\mathcal{C}, \otimes, I, a, c, l, r, U, t)$ be an IBMC. Let $g \in \exp \mathfrak{t}_3^0$. Set

$$\tilde{a}_{XYZ} := a_{XYZ} \cdot g(a_{XYZ}^{-1}(t_{XY} \otimes id_Z)a_{XYZ}, id_X \otimes t_{YZ})^{-1}.$$

Then the set $\tilde{\mathbb{C}} = (\mathcal{C}, \otimes, I, \tilde{a}, c, l, r, U, t)$ is an IBMC if and only if $g \in \text{GRT}_1(\mathbf{k})$, i.e. it satisfies (1.5)-(1.7) (c.f. [Dr]). This yields that $\text{GRT}_1(\mathbf{k})$ forms a group by (1.8).

Let $g \in \text{GRT}_1(\mathbf{k})$ and $h \in \exp \mathfrak{t}_{3,N}^0$. Let $\mathbb{C} = (\mathcal{C}, \otimes, I, a, c, l, r, U, t, \sigma)$ be a C_N -IBMC and $\mathbb{M} = (\mathcal{M}, \otimes, b, r, V, t)$ be an IMC over it. Define $\tilde{\mathbb{C}}$ as above. Put $\tilde{\mathbb{M}} = (\mathcal{M}, \otimes, \tilde{b}, r, V, t)$ with

$$\begin{aligned} \tilde{b}_{MXY} &= b_{MXY} \cdot h(b_{MXY}^{-1}(t_{MX} \otimes id_Y)b_{MXY}, id_M \otimes t_{XY}, \\ &\quad b_{MXY}^{-1}(id_{M \otimes X} \otimes \sigma_Y)b_{MXY}(id_M \otimes t_{XY}), \dots, \\ &\quad \dots, b_{MXY}^{-1}(id_{M \otimes X} \otimes \sigma_Y^{N-1})b_{MXY}(id_M \otimes t_{XY}))^{-1}. \end{aligned}$$

Lemma A.1. *The new set $\tilde{\mathbb{M}}$ is an IMC over $\tilde{\mathbb{C}}$ if and only if $(g, h) \in \text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})$,*

Proof. The equation (2.5), (2.6) and (2.7) guarantee respectively the mixed pentagon axiom in (I), the octagon axiom in (II) and the first equality in (IV). As for the second equality in (IV), it is automatic because $t_{MX} \otimes id_Y$ and $(id_{M \otimes X} \otimes \sigma_Y^a)b_{MXY}(id_M \otimes t_{XY})b_{MXY}^{-1}$ ($a \in C_N$) commute with $t_{M \otimes X, Y} + t_{MX} \otimes id_Y$.

Conversely by taking $(\mathbb{C}, \mathbb{M}) = (\mathbb{C}_{\text{univ}}, \mathbb{M}_{\text{univ}})$, one sees that the presentations (I)-(IV) imply the relations (2.5)-(2.7). \square

As a corollary we get

Proposition A.2. *The set $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})$ forms a group by the multiplication (2.8).*

For $n \geq 1$ define $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})^{(n)}$ to be the set of $(g, h) \pmod{\deg n} \in \exp \mathfrak{t}_3^0 \times \exp \mathfrak{t}_{3,N}^0 \pmod{\deg n}$ which satisfies all the defining equations of $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})$ modulo $\deg n$. By considering all IMC \mathbb{M} over C_N -IBMC \mathbb{C} such that $\Gamma^{n+1}U_X = \Gamma^{n+1}V_M = \{1\}$ (Γ^n : the n -th term of lower central series) holds for any $X \in \text{Ob}\mathbb{C}$ and $M \in \text{Ob}\mathbb{M}$, we see that $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})^{(n)}$ forms an algebraic group. The following was required to prove Theorem 3.4.

Lemma A.3. *The natural morphism $\text{GRTM}_{(\bar{1},1)}(N, \mathbf{k}) \rightarrow \text{GRTM}_{(\bar{1},1)}(N, \mathbf{k})^{(n)}$ is surjective.*

Proof. This is a morphism of pro-unipotent algebraic group, which induces a Lie algebra morphism

$$\mathbf{grtm}_{(\mathbb{I},1)}(N, \mathbf{k}) = \prod_{k=1}^{\infty} \mathbf{grtm}_{(\mathbb{I},1)}(N, \mathbf{k})^{(k)} \rightarrow \mathbf{grtm}_{(\mathbb{I},1)}^{(n)}(N, \mathbf{k}) = \oplus_{k=1}^n \mathbf{grtm}_{(\mathbb{I},1)}(N, \mathbf{k})^{(k)}.$$

Here $\mathbf{grtm}_{(\mathbb{I},1)}(N, \mathbf{k})^{(k)}$ means the degree k -component of $\mathbf{grtm}_{(\mathbb{I},1)}(N, \mathbf{k})$. Since the Lie algebra morphism is surjective, so is the pro-algebraic group morphism. \square

APPENDIX B. ERRATUM OF [E]

We use this opportunity to correct some errors in [E].

- (1) The first two formulae in [E] page 400 should be replaced by

$$\Phi_{KZ}^{0,1,23} \Phi_{KZ}^{01,2,3} = \Phi_{KZ}^{1,2,3} \Phi_{KZ}^{0,12,3} \Phi_{KZ}^{0,1,2}$$

and

$$\Psi_{KZ}^{0,1,23} \Psi_{KZ}^{01,2,3} = \Phi_{KZ}^{1,2,3} \Psi_{KZ}^{0,12,3} \Psi_{KZ}^{0,1,2}.$$

- (2) For a ring R and $N \geq 1$ the definition of the ring $R(N)$ in [E]§6.2 should be read as follows³: $R(N) = (\mathbf{Z}/N\mathbf{Z}) \times R$, with the following operations. The sum is given by

$$(a, r) + (a', r') = (a + a', r + r' + \sigma(a, a'))$$

and the product is given by

$$(a, r)(a', r') = (aa', \tilde{a}r' + \tilde{a}'r + Nrr' + \pi(a, a')),$$

where $a \mapsto \tilde{a}$ is the map $(\mathbf{Z}/N\mathbf{Z}) \rightarrow \{0, 1, \dots, N-1\}$ inverse to the ‘reduction modulo N ’ map, and $\sigma, \pi : (\mathbf{Z}/N\mathbf{Z})^2 \rightarrow \mathbf{Z}$ are defined by $\widetilde{a + a'} = \tilde{a} + \tilde{a'} + N\sigma(a, a')$, $\widetilde{aa'} = \tilde{a}\tilde{a'} + N\pi(a, a')$.

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³The first author is grateful to I. Marin for pointing out the inconsistency of the definition in [E].

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